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WEAKENING OF AN ELASTIC SOLID BY A PERIODIC ARRAY OF PENNY-SHAPED CRACKS†

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Abstract—The three-dimensional stress distribution for a periodic array of penny-shaped cracks in an infinite isotropic elastic solid under arbitrary uniform loadings is considered by means of the Somigliana dislocation method. The displacement discontinuity of the Somigliana dislocations, by which the penny-shaped cracks are simulated mathematically, is assumed to be in the form $p(x_1', x_2')(1-x_1'^2-x_2'^2)^{1/2}$. The unknown polynomial $p(x_1', x_2')$ is determined from the boundary conditions on the surfaces of the penny-shaped cracks. It is shown that the singular behavior of the three-dimensional stresses in the vicinity of the crack border is easily examined and the stress intensity factors can be expressed explicitly as functions of position along the crack border. Numerical results for the stress intensity factors are shown for the case of the periodic array of coplanar penny-shaped cracks.

1. INTRODUCTION

In studying the fracture strength of materials, considerable effort has been made in the investigation of the three-dimensional problems involving penny-shaped and elliptical cracks. However, the majority of the literature[1–6] on this subject has been concerned with a single crack. In general, engineering materials contain numerous cracks. Therefore it is also important to consider the interaction problems for penny-shaped cracks and elliptical cracks.

The first analytical work on the three-dimensional interaction problems for cracks dates back to 1962, when Collins[7] treated infinite elastic solids containing two symmetrically located penny-shaped cracks and an infinite row of equally spaced parallel cracks of equal radii. He also discussed the opening by applied pressures of two coplanar penny-shaped cracks and an infinite row of coplanar penny-shaped cracks[8]. In these works, the problems were reduced to the solutions of the Fredholm integral equations of the second kind with the use of potential functions, but no numerical calculations were carried out. Recently, Watanabe and Atsumi[9] have discussed an axially symmetric problem of a long circular cylinder containing an infinite row of parallel penny-shaped cracks.

The present paper deals with the three-dimensional crack problem for a periodic array of penny-shaped cracks in an infinite isotropic elastic solid under arbitrary uniform loadings. To solve this problem, the Somigliana dislocation method is adopted, that is, the penny-shaped cracks are simulated by the Somigliana dislocations in the same shape. The displacement discontinuity of the Somigliana dislocations is assumed to be in the form $p(x_1', x_2')(1-x_1'^2-x_2'^2)^{1/2}$ where $p(x_1', x_2')$ is a polynomial in the local coordinate system x_1' and x_2' taken at the center of each Somigliana dislocation and is determined from the boundary conditions. By means of this method, the singular behavior of the three-dimensional stresses near the crack border is easily examined and the stress intensity factors can be expressed explicitly as functions of position along the crack border. Numerical results for the stress intensity factors are computed for the case of the periodic array of coplanar penny-shaped cracks, and they are plotted in terms of the geometrical parameters of crack arrangement.

2. BASIC FORMULAS OF SOMIGLIANA DISLOCATION THEORY

The three-dimensional elastic field around an elliptical Somigliana dislocation in an infinite anisotropic elastic medium has been considered in a previous paper[10]. In this section, only the results are reviewed briefly. When the elliptical Somigliana dislocation, of which the displacement discontinuity is b_m, lies in the x_3 plane of a Cartesian coordinate system x_k, the

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displacement $u_i(\mathbf{x})$ and displacement gradients $u_{i,j}(\mathbf{x})$ are given by

$$u_i(\mathbf{x}) = -\frac{1}{8\pi^2}\int_{S^*} \mathrm{d}S(\bar{\boldsymbol{\zeta}})\left[\frac{\partial}{\partial z}\int_{-R}^{R} b_m(\mathbf{x}')\,\mathrm{d}t\right]_{z=\bar{\boldsymbol{\zeta}}\cdot \mathbf{y}/\sin\theta} C_{klm3}N_{ik}(\bar{\boldsymbol{\xi}})D^{-1}(\bar{\boldsymbol{\xi}})\zeta\bar{\xi}_l/\sin^2\theta,$$

$$u_{i,j}(\mathbf{x}) = -\frac{1}{8\pi^2}\int_{S^*} \mathrm{d}S(\bar{\boldsymbol{\zeta}})\left[\frac{\partial}{\partial z}\int_{-R}^{R} b_m(\mathbf{x}')\,\mathrm{d}t\right]_{z=\bar{\boldsymbol{\zeta}}\cdot \mathbf{y}/\sin\theta} (\partial/\partial\theta)[C_{klm3}N_{ik}(\bar{\boldsymbol{\xi}})D^{-1}(\bar{\boldsymbol{\xi}})\bar{\xi}_j\bar{\xi}_l]y_3^{-1}/\sin\theta \tag{1}$$

where C_{klmn} are the elastic moduli, $N_{ik}(\bar{\boldsymbol{\xi}})$ and $D(\bar{\boldsymbol{\xi}})$ are, respectively, the cofactor and the determinant of the 3×3 matrix with elements $C_{ipkq}\bar{\xi}_p\bar{\xi}_q$,† z and t are the coordinate variables defined on the plane of the Somigliana dislocation, $R = (1-z^2)^{1/2}$, and $\bar{\boldsymbol{\zeta}}$ is the unit vector expressed by the spherical polar coordinates in the form

$$\bar{\zeta}_1 = \cos\phi\sin\theta,\ \bar{\zeta}_2 = \sin\phi\sin\theta,\ \bar{\zeta}_3 = \cos\theta. \tag{2}$$

The surface integrals in (1) are performed on S^* which is the subdomain of the unit sphere $\bar{\zeta}_i\bar{\zeta}_i = 1$ satisfying the condition:

$$|\bar{\boldsymbol{\zeta}}\cdot \mathbf{y}/\sin\theta| \le 1. \tag{3}$$

If the shape of the Somigliana dislocation is of circular disk with unit radius, we obtain the following relations:

$$x_i = y_i,\ \bar{\xi}_i = \bar{\zeta}_i,\ \zeta = (\zeta_i\zeta_i)^{1/2} = 1. \tag{4}$$

For isotropic media, we have simple expressions for C_{klmn} and $N_{ik}(\bar{\boldsymbol{\xi}})D^{-1}(\bar{\boldsymbol{\xi}})$, i.e.

$$C_{klmn} = \lambda\delta_{kl}\delta_{mn} + \mu\delta_{km}\delta_{ln} + \mu\delta_{kn}\delta_{lm},$$

$$N_{ik}(\bar{\boldsymbol{\xi}})D^{-1}(\bar{\boldsymbol{\xi}}) = \frac{(\lambda+2\mu)\delta_{ik} - (\lambda+\mu)\bar{\xi}_i\bar{\xi}_k}{\mu(\lambda+2\mu)} \tag{5}$$

where λ and μ are Lamé's elastic constants and δ_{kl} is Kronecker's delta.

Let us consider the displacement discontinuity $b_m(\mathbf{x}')$ of the penny-shaped Somigliana dislocation given by

$$b_m(\mathbf{x}') = A_m^{(M_1,M_2)}x_1'^{M_1}x_2'^{M_2}(1 - x_1'^2 - x_2'^2)^{1/2} \tag{6}$$

where $A_m^{(M_1,M_2)}$ are constants.‡ Using the binomial theorem and the integral formula associated with $(R^2 - t^2)^{1/2}$, we obtain

$$\frac{\partial}{\partial z}\int_{-R}^{R} b_m(\mathbf{x}')\,\mathrm{d}t = A_m^{(M_1,M_2)}\sum_{K_1,K_2=0}^{M_1,M_2}\ \sum_{J=0}^{(K/2)+1}\binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\frac{\pi}{2}Q(K)$$
$$\times(-1)^{(K/2)-J+1}(M-2J+2)m_1^{M_1-K_1}m_2^{M_2-K_2}n_1^{K_1}n_2^{K_2}z^{M-2J+1} \tag{7}$$

where

$$K = K_1 + K_2,\ M = M_1 + M_2,$$

$$Q(K) = \begin{cases} \dfrac{(K-1)(K-3)\ldots 3\cdot 1}{(K+2)(K+4)\ldots 2K} & (K\ \text{even}) \\ 0 & (K\ \text{odd}) \\ 1 & (K=0), \end{cases} \tag{8}$$

†Latin indices range over 1, 2, 3, and the usual summation convention is applied to every repeated index.
‡The summation convention is not applied to the indices M_1 and M_2.

$$m_\alpha = \begin{cases} \cos\phi & (\alpha = 1) \\ \sin\phi & (\alpha = 2) \end{cases}, \qquad n_\alpha = \begin{cases} -\sin\phi & (\alpha = 1) \\ \cos\phi & (\alpha = 2) \end{cases}.$$

By substituting (7) into the second expression of (1) and using

$$\bar{\zeta}\cdot \mathbf{y}/\sin\theta = x_1\cos\phi + x_2\sin\phi + x_3\cos\theta/\sin\theta, \tag{9}$$

the displacement gradients are written as

$$\begin{aligned} u_{i,j}(\mathbf{x}) = -A_m^{(M_1,M_2)} & \sum_{K_1,K_2=0}^{M_1,M_2} \sum_{J=0}^{(K/2)+1} \binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\frac{1}{16\pi}Q(K)(-1)^{(K/2)+K_1-J+1} \\ & \times (M-2J+2)\int_0^{2\pi} \mathrm{d}\phi \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \,[(\cos\phi)^{M_1-K_1+K_2}(\sin\phi)^{M_2+K_1-K_2} \\ & \times (x_1\cos\phi + x_2\sin\phi + x_3\cos\theta/\sin\theta)^{M-2J+1}x_3^{-1}(\partial\Gamma_{ijm}/\partial\theta)] \end{aligned} \tag{10}$$

where

$$\Gamma_{ijm} = \frac{[\lambda\delta_{kl}\delta_{m3} + \mu(\delta_{km}\delta_{l3} + \delta_{k3}\delta_{lm})][(\lambda+2\mu)\delta_{ik} - (\lambda+\mu)\bar{\zeta}_i\bar{\zeta}_k]\bar{\zeta}_j\bar{\zeta}_l}{\mu(\lambda+2\mu)} \tag{11}$$

and

$$\begin{aligned} \theta_1 &= \tan^{-1}[x_3/(1 - x_1\cos\phi - x_2\sin\phi)], \\ \theta_2 &= \tan^{-1}[-x_3/(1 + x_1\cos\phi + x_2\sin\phi)]. \end{aligned} \tag{12}$$

When x_3 becomes zero, the displacement gradients $u_{i,j}(\mathbf{x})$ are simplified as follows:

Case (1). $x < 1$ where $x = (x_i x_i)^{1/2}$.
In this case, we have

$$\theta_1 = 0,\ \theta_2 = \pi. \tag{13}$$

In view of the relation:

$$\int_0^\pi (\partial\Gamma_{ijm}/\partial\theta)\,\mathrm{d}\theta = 0, \tag{14}$$

the displacement gradients on the plane of the penny-shaped Somigliana dislocation are given by

$$\begin{aligned} u_{i,j}(\mathbf{x}) = -A_m^{(M_1,M_2)} & \sum_{K_1,K_2=0}^{M_1,M_2} \sum_{J=0}^{(K/2)+1} \binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\frac{1}{16\pi}Q(K)(-1)^{(K/2)+K_1-J+1} \\ & \times (M-2J+2)(M-2J+1)\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \,[(\cos\phi)^{M_1-K_1+K_2}(\sin\phi)^{M_2+K_1-K_2} \\ & \times (x_1\cos\phi + x_2\sin\phi)^{M-2J}(\cos\theta/\sin\theta)(\partial\Gamma_{ijm}/\partial\theta)]. \end{aligned} \tag{15}$$

Case (2). $x > 1$.
Expression (10) can be reduced to

$$\begin{aligned} u_{i,j}(\mathbf{x}) = -A_m^{(M_1,M_2)} & \sum_{K_1,K_2=0}^{M_1,M_2} \sum_{J=0}^{(K/2)+1} \binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\frac{1}{16\pi}Q(K)(-1)^{(K/2)+K_1-J+1} \\ & \times (M-2J+2)\lim_{x_3\to 0}\int_0^{2\pi} \mathrm{d}\phi \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \,\{(\cos\theta)^{M_1-K_1+K_2} \\ & \times (\sin\phi)^{M_2+K_1-K_2}[(x_1\cos\phi + x_2\sin\phi)^{M-2J+1}x_3^{-1} \\ & + (M-2J+1)(x_1\cos\phi + x_2\sin\phi)^{M-2J}(\cos\theta/\sin\theta)](\partial\Gamma_{ijm}/\partial\theta)\}. \end{aligned} \tag{16}$$

3. STATEMENT OF THE PROBLEM

Considerations will be given to the three-dimensional stress distribution for a periodic array of penny-shaped cracks, whose radii are 1, in an infinite isotropic elastic solid under arbitrary uniform loadings. It is supposed that the surface of every penny-shaped crack is free from tractions. The origin of a Cartesian coordinate system x_k is located at the center of one of the penny-shaped cracks, as shown in Fig. 1. The ratios of periods of the cracks in the directions of the x_1,x_2 and x_3 axes to the radius of the cracks are denoted by a_1, a_2 and a_3, respectively.

4. REDUCTION TO LINEAR ALGEBRAIC EQUATIONS

To solve the problem, every penny-shaped crack is simulated by the Somigliana dislocation in the same shape. The periodicity of the cracks and the uniformity of the applied stresses require the same distributions of displacement discontinuity of the Somigliana dislocations. As a consequence, the considerations of this problem can be given in $|x_i| \leq a_i/2$. The displacement discontinuity of the Somigliana dislocations is assumed to be in the form $p(x_1', x_2')(1 - x_1'^2 - x_2'^2)^{1/2}$ where $p(x_1', x_2')$ is a polynomial in the local coordinate system x_1' and x_2' taken at the center of each Somigliana dislocation. Then, in view of (10), we obtain the displacement gradients in the following form:

$$
\begin{aligned}
u_{i,j}(\mathbf{x}) = &- \sum_{M_1,M_2=0}^{\infty} A_m^{(M_1,M_2)} \sum_{K_1,K_2=0}^{M_1,M_2} \sum_{J=0}^{(K/2)+1} \binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J} \frac{1}{16\pi} Q(K) \\
&\times (-1)^{(K/2)+K_1-J+1}(M-2J+2) \sum_{L_1,L_2,L_3=-\infty}^{\infty} \int_0^{2\pi} \mathrm{d}\phi \int_{\theta_1^*}^{\theta_2^*} \mathrm{d}\theta \\
&\times \{(\cos\phi)^{M_1-K_1+K_2}(\sin\phi)^{M_2+K_1-K_2}[(x_1+L_1a_1)\cos\phi + (x_2+L_2a_2)\sin\phi \\
&+ (x_3+L_3a_3)\cos\theta/\sin\theta]^{M-2J+1}(x_3+L_3a_3)^{-1}(\partial\Gamma_{ijm}/\partial\theta)\}
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
\theta_1^* &= \tan^{-1}\{(x_3+L_3a_3)/[1-(x_1+L_1a_1)\cos\phi - (x_2+L_2a_2)\sin\phi]\}, \\
\theta_2^* &= \tan^{-1}\{-(x_3+L_3a_3)/[1+(x_1+L_1a_1)\cos\phi + (x_2+L_2a_2)\sin\phi]\}.
\end{aligned}
\tag{18}
$$

The corresponding stresses can be obtained by substituting (17) into Hooke's law for isotropic media:

$$
\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{19}
$$

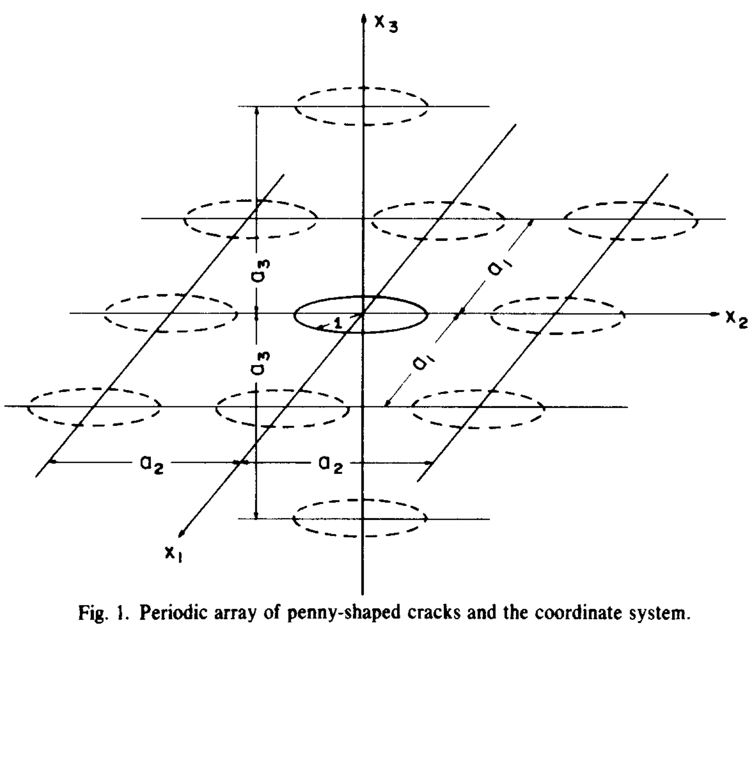

Fig. 1. Periodic array of penny-shaped cracks and the coordinate system.

Since the surface of every penny-shaped crack is free from tractions, the boundary conditions on the crack surface can be written as

$$\sigma_{3i}(\mathbf{x}) + \sigma_{3i}^{A} = 0 \tag{20}$$

where σ_{3i}^{A} denotes a uniform applied stress.

For numerical calculation the infinite series (17) are truncated after $M_1 + M_2$ reaches a finite number N. Then, through (19) the stress components on the plane of the Somigliana dislocation, i.e. $x < 1$ and $x_3 = 0$, are expressed as

$$\sigma_{3i}(\mathbf{x}) = \sum_{M_1, M_2=0}^{M_1+M_2=N} A_m^{(M_1, M_2)} F_{im}(\mathbf{x}). \tag{21}$$

Substituting (21) into (20) and using the boundary collocation technique, we obtain

$$\sum_{M_1, M_2=0}^{M_1+M_2=N} A_m^{(M_1, M_2)} F_{im}(\mathbf{x}^{(P)}) + \sigma_{3i}^{A} = 0 \tag{22}$$

where $\mathbf{x}^{(P)}$ are arbitrarily selected collocation points which satisfy $x_i^{(P)} x_i^{(P)} < 1$ and $x_3^{(P)} = 0$. Once the coefficients $A_m^{(M_1, M_2)}$ have been determined from the linear algebraic equations (22), the three-dimensional stresses for a periodic array of penny-shaped cracks in an infinite elastic solid can be calculated.

5. STRESS INTENSITY FACTORS

The stress intensity factors for a penny-shaped crack and an elliptical crack in infinite elastic solids subjected to normal and shear loadings have been considered by several investigators [1–6]. In this section we shall give the expression of the stress intensity factors for a periodic array of penny-shaped cracks on the basis of the Somigliana dislocation theory. In view of (16), the singular behavior of the displacement gradients at the crack border ($x = 1$, $x_3 = 0$) is given by

$$\begin{aligned} u_{i,j}(\mathbf{x}) = &- \sum_{M_1, M_2=0}^{\infty} A_m^{(M_1, M_2)} \sum_{K_1, K_2=0}^{M_1, M_2} \sum_{J=0}^{(K/2)+1} \binom{M_1}{K_1} \binom{M_2}{K_2} \binom{(K/2)+1}{J} \frac{1}{16\pi} Q(K) \\ &\times (-1)^{(K/2)+K_1-J+1} (M - 2J + 2) \lim_{x\to 1} \lim_{x_3 \to 0} \int_0^{2\pi} \mathrm{d}\phi \{(\cos\phi)^{M_1-K_1+K_2} \\ &\times (\sin\phi)^{M_2+K_1-K_2} (x_1 \cos\phi + x_2 \sin\phi)^{M-2J+1} x_3^{-1} [\Gamma_{ijm}]_{\theta_1}^{\theta_2} \}. \end{aligned} \tag{23}$$

Here we note the following integral formulas:

$$\begin{aligned} &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \cos^2\theta_1 x_3^{-1}\, \mathrm{d}\phi = -\frac{2\pi x_1^p x_2^q}{(x^2-1)^{1/2}} \quad (p+q \text{ odd}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \cos^2\theta_2 x_3^{-1}\, \mathrm{d}\phi = \frac{2\pi x_1^p x_2^q}{(x^2-1)^{1/2}} \quad (p+q \text{ odd}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \sin\theta_1 \cos\theta_1 x_3^{-1}\, \mathrm{d}\phi = O(1) \quad (p,q \text{ non-negative integer}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \sin\theta_2 \cos\theta_2 x_3^{-1}\, \mathrm{d}\phi = O(1) \quad (p,q \text{ non-negative integer}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \cos^4\theta_1 x_3^{-1}\, \mathrm{d}\phi = -\frac{3\pi x_1^p x_2^q}{(x^2-1)^{1/2}} \quad (p+q \text{ odd}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \cos^4\theta_2 x_3^{-1}\, \mathrm{d}\phi = \frac{3\pi x_1^p x_2^q}{(x^2-1)^{1/2}} \quad (p+q \text{ odd}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \sin\theta_1 \cos^3\theta_1 x_3^{-1}\, \mathrm{d}\phi = O(1) \quad (p,q \text{ non-negative integer}), \\ &\lim_{x\to 1} \lim_{x_3\to 0} \int_0^{2\pi} (\cos\phi)^p (\sin\phi)^q \sin\theta_2 \cos^3\theta_2 x_3^{-1}\, \mathrm{d}\phi = O(1) \quad (p,q \text{ non-negative integer}) \end{aligned} \tag{24}$$

where θ_1 and θ_2 are defined in (12). These integral formulas can be obtained by introducing the complex variable $\eta = e^{i\phi}$ and calculating the residues in the unit circle on the η plane.

Substituting (23) into (19) and using (24), we obtain, after some manipulations, the stress components σ_{3i} on the plane $x_3 = 0$ near the crack border, as follows:

$$\begin{Bmatrix}\sigma_{31}\\ \sigma_{32}\\ \sigma_{33}\end{Bmatrix} = -\frac{\mu}{4(x^2-1)^{1/2}}\sum_{M_1,M_2=0}^{\infty} A_m^{(M_1,M_2)} \sum_{K_1,K_2=0}^{M_1,M_2}\sum_{J=0}^{(K/2)+1}\sum_{I=0}^{M-2J+1}\binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\binom{M-2J+1}{I}$$

$$\times Q(K)(-1)^{(K/2)+K_1-J+1}(M-2J+2)x_1^{M_1-K_1+K_2+2(M-2J-I+1)}x_2^{M_2+K_1-K_2+2I}\begin{Bmatrix}\Delta_m\\ \Delta_m'\\ \Delta_m''\end{Bmatrix} \tag{25}$$

where

$$\Delta_1 = 1+\frac{\lambda}{\lambda+2\mu}x_1^2,\ \Delta_2 = \frac{\lambda}{\lambda+2\mu}x_1x_2,\ \Delta_3 = 0,$$
$$\Delta_1' = \frac{\lambda}{\lambda+2\mu}x_1x_2,\ \Delta_2' = 1+\frac{\lambda}{\lambda+2\mu}x_2^2,\ \Delta_3' = 0, \tag{26}$$
$$\Delta_1'' = \Delta_2'' = 0,\ \Delta_3'' = \frac{2(\lambda+\mu)}{\lambda+2\mu}.$$

By introducing the cylindrical coordinates:

$$x_1 = \rho\cos\omega,\ x_2 = \rho\sin\omega,\ x_3 = Z, \tag{27}$$

the stress intensity factors for the penny-shaped crack are defined in terms of the stress components σ_{ZZ}, $\sigma_{Z\rho}$ and $\sigma_{Z\omega}$ on the plane $Z = 0$ as

$$\sigma_{ZZ} = k_1/(2r)^{1/2},\ \sigma_{Z\rho} = k_2/(2r)^{1/2},\ \sigma_{Z\omega} = k_3/(2r)^{1/2} \tag{28}$$

where r is a radial distance from the crack border, and k_1, k_2 and k_3 are, respectively, the stress intensity factors for the opening, edge-sliding and tearing modes. From (25) the stress intensity factors of this problem are given by

$$\begin{Bmatrix}k_1\\ k_2\\ k_3\end{Bmatrix} = -\frac{\mu}{4}\sum_{M_1,M_2=0}^{\infty} A_m^{(M_1,M_2)} \sum_{K_1,K_2=0}^{M_1M_2}\sum_{J=0}^{(K/2)+1}\sum_{I=0}^{M-2J+1}\binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\binom{M-2J+1}{I}$$
$$\times Q(K)(-1)^{(K/2)+K_1-J+1}(M-2J+2)(\cos\omega)^{M_1-K_1+K_2+2(M-2J-I+1)}$$
$$\times(\sin\omega)^{M_2+K_1-K_2+2I}\begin{Bmatrix}\Lambda_m\\ \Lambda_m'\\ \Lambda_m''\end{Bmatrix} \tag{29}$$

where

$$\Lambda_1 = \Lambda_2 = 0,\ \Lambda_3 = \frac{2(\lambda+\mu)}{\lambda+2\mu},$$
$$\Lambda_1' = \frac{2(\lambda+\mu)}{\lambda+2\mu}\cos\omega,\ \Lambda_2' = \frac{2(\lambda+\mu)}{\lambda+2\mu}\sin\omega,\ \Lambda_3' = 0, \tag{30}$$
$$\Lambda_1'' = -\sin\omega,\ \Lambda_2'' = \cos\omega,\ \Lambda_3'' = 0.$$

Here it should be noted that once the coefficients $A_m^{(M_1,M_2)}$ of the displacement discontinuity have been determined, the stress intensity factors can be readily evaluated from (29) for arbitrary positions of the crack border.

Let us here consider a particular case where a penny-shaped crack, whose radius is 1, is situated in an infinite isotropic solid under arbitrary uniform loadings. In this case, we have

$$a_1 = \infty,\ a_2 = \infty,\ a_3 = \infty. \tag{31}$$

Substituting (31) into (17) and in view of (15), we obtain the displacement gradients on the plane of the Somigliana dislocation, which corresponds to the penny-shaped crack:

$$\begin{aligned} u_{i,j}(\mathbf{x}) = &- \sum_{M_1,M_2=0}^{\infty} A_m{}^{(M_1,M_2)} \sum_{K_1,K_2=0}^{M_1,M_2} \sum_{J=0}^{(K/2)+1} \binom{M_1}{K_1}\binom{M_2}{K_2}\binom{(K/2)+1}{J}\frac{1}{16\pi}Q(K) \\ &\times(-1)^{(K/2)+K_1-J+1}(M-2J+2)(M-2J+1) \\ &\times \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta[(\cos\phi)^{M_1-K_1+K_2}(\sin\phi)^{M_2+K_1-K_2} \\ &\times (x_1\cos\phi + x_2\sin\phi)^{M-2J}(\cos\theta/\sin\theta)(\partial\Gamma_{ijm}/\partial\theta)]. \end{aligned} \tag{32}$$

By the use of the theorem [10] concerning the stresses on the plane of the elliptical Somigliana dislocation, the uniformity of the applied stresses at the infinity in the infinite solid yields

$$A_m{}^{(M_1,M_2)} = 0 \quad (M_1 + M_2 > 0). \tag{33}$$

Then, (32) becomes

$$u_{i,j} = S_{ijm}A_m{}^{(0,0)} \tag{34}$$

where

$$\begin{aligned} &S_{131} = S_{232} = -\frac{\pi}{8}\frac{3\lambda+5\mu}{\lambda+2\mu}, \quad S_{311} = S_{322} = \frac{\pi}{8}\frac{\mu}{\lambda+2\mu}, \\ &S_{113} = S_{223} = -\frac{\pi}{8}\frac{\mu}{\lambda+2\mu}, \quad S_{333} = -\frac{\pi}{4}\frac{\mu}{\lambda+2\mu} \end{aligned} \tag{35}$$

and the coefficients S_{ijm} which are not given in (35) are to be zero.

When a penny-shaped crack is situated in an infinite elastic solid subjected to a uniform tension σ_{33}^A perpendicular to the crack plane, substitution of (34) into (20) leads to

$$A_1{}^{(0,0)} = A_2{}^{(0,0)} = 0, \quad A_3{}^{(0,0)} = \frac{2(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\sigma_{33}^A. \tag{36}$$

By substituting (33) and (36) into (29), the stress intensity factors for a penny-shaped crack in an infinite solid under a uniform tension σ_{33}^A are given by

$$k_1 = \frac{2}{\pi}\sigma_{33}^A, \quad k_2 = k_3 = 0 \tag{37}$$

which agrees with the result in [1].

If the infinite solid containing a penny-shaped crack is subjected to a uniform shear σ_{13}^A parallel to the crack plane, we get

$$A_1{}^{(0,0)} = \frac{8(\lambda+2\mu)}{\pi\mu(3\lambda+4\mu)}\sigma_{13}^A, \quad A_2{}^{(0,0)} = A_3{}^{(0,0)} = 0. \tag{38}$$

Then, the stress intensity factors are written as

$$\begin{aligned} &k_1 = 0,\ k_2 = \frac{8(\lambda+\mu)}{\pi(3\lambda+4\mu)}\sigma_{13}^A\cos\omega = \frac{4}{\pi(2-\nu)}\sigma_{13}^A\cos\omega, \\ &k_3 = -\frac{4(\lambda+2\mu)}{\pi(3\lambda+4\mu)}\sigma_{13}^A\sin\omega = -\frac{4(1-\nu)}{\pi(2-\nu)}\sigma_{13}^A\sin\omega \end{aligned} \tag{39}$$

where ν is Poisson's ratio. The same result can be derived from the paper of Kassir and Sih[2].

6. NUMERICAL EXAMPLES AND DISCUSSIONS

The aforementioned method will be applied to the study of influence of some geometrical parameters of crack arrangement on the stress intensity factors. Numerical calculations are performed for a periodic array of coplanar penny-shaped cracks with unit radius which are situated in an infinite isotropic solid under σ_{33}^A and σ_{31}^A (Fig. 1). The value of Poisson's ratio is taken as $\nu = 0.3$. The stress intensity factors are shown as Figs. 2–5. In these figures, k_{10} is the stress intensity factor for the opening mode at the border of a penny-shaped crack under a uniform tension σ_{33}^A, and k_{20} and k_{30} are, respectively, the stress intensity factors for the edge-sliding mode at $\omega = 0$ and tearing mode at $\omega = \pi/2$ under a uniform shear σ_{13}^A. From (37) and (39), k_{10}, k_{20} and k_{30} are given by

$$k_{10} = \frac{2}{\pi}\sigma_{33}^A, \quad k_{20} = \frac{4}{\pi(2-\nu)}\sigma_{13}^A, \quad k_{30} = -\frac{4(1-\nu)}{\pi(2-\nu)}\sigma_{13}^A. \tag{40}$$

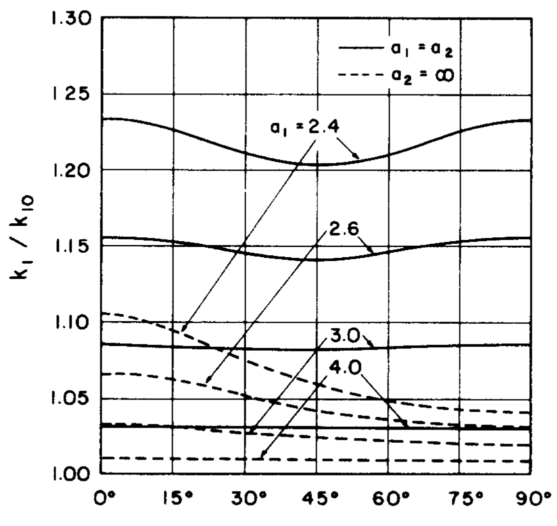

Fig. 2. The stress intensity factor for the opening mode vs the position along the crack border of coplanar cracks ($a_3 = \infty$).

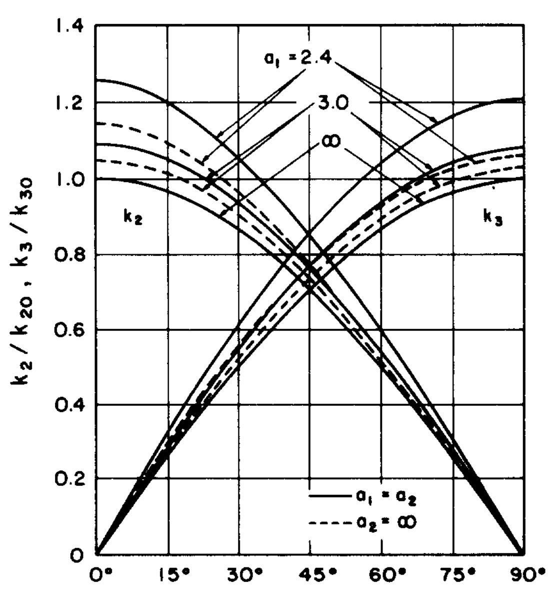

Fig. 3. The stress intensity factors for the edge-sliding and tearing modes vs the position along the crack border for $a_1 = a_2$ and $a_3 = \infty$.

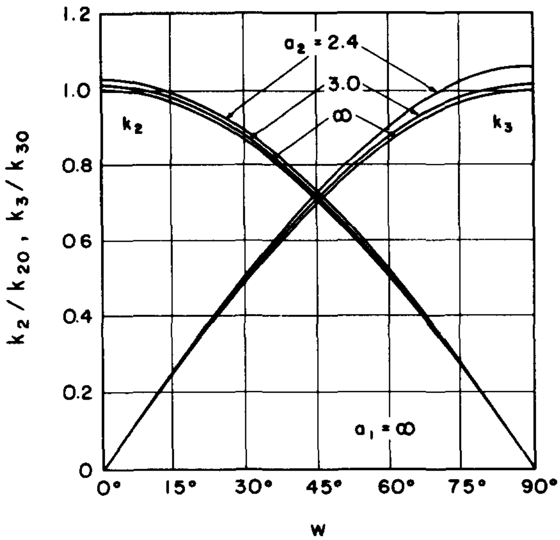

Fig. 4. The stress intensity factors for the edge-sliding and tearing modes vs the position along the crack border for $a_1 = \infty$.

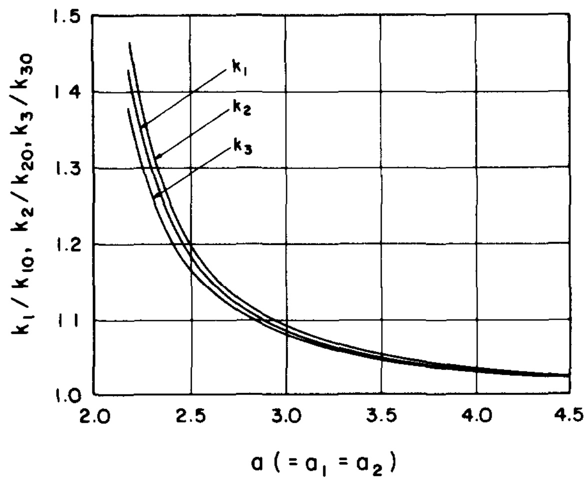

Fig. 5. The maximum stress intensity factors vs the periods of the penny-shaped cracks.

Figure 2 shows the variation of the stress intensity factor for the opening mode against the position along the crack border for various periods of cracks. The medium is subjected to a uniform tension σ_{33}^A perpendicular to the crack planes. The full and broken lines indicate, respectively, the results for the cases of $a_1 = a_2$ and $a_2 = \infty$. The case of $a_2 = \infty$ means the infinite row of coplanar penny-shaped cracks whose centers lie on the x_1 axis. It is seen from this figure that for $a_1 = a_2$, the stress intensity factor k_1 takes the maximum value at the directions of the x_1 and x_2 axes and has the minimum one at the angle 45° with respect to x_1 axis. On the other hand, for $a_2 = \infty$ the stress intensity factor becomes maximum at the direction of the x_1 axis and minimum at that of the x_2 axis. This figure also reveals that the stress intensity factor k_1 increases with the shortening of the periods of the penny-shaped cracks. When the medium undergoes a uniform shear σ_{13}^A parallel to the crack planes, the stress intensity factors for the edge-sliding and tearing modes are plotted vs the position along the crack border in Figs. 3 and 4. Examination of these figures shows that for a row of cracks ($a_1 = \infty$) perpendicular to the direction of the applied shear stress, the dependency of the stress intensity factors on the period of crack arrangement is insensible. If the periods of the cracks are sufficiently large compared with the radii, i.e. $a_1 = \infty$ and $a_2 = \infty$, the stress intensity factors k_2 and k_3 are given from (39) by $k_2/k_{20} = \cos\omega$ and $k_3/k_{30} = \sin\omega$. For the case of $a_1 = a_2$, the maximum values of the stress intensity factors k_1, k_2 and k_3 vs the periods of the cracks are shown in Fig. 5. It can be recognized that the maximum stress intensity factors increase with the shortening of the periods.

Finally the accuracy of the numerical results will be mentioned. The approximate solution is obtained by truncating the series (17) for $M_1 + M_2 > N$. The computation reveals that the convergence of the series (17) becomes slower as the periods of the penny-shaped cracks are shorter. For the case of $a_1 = a_2$, we have examined the speed of convergence. As a consequence, it is found that the result for $N = 8$ gives satisfactory accuracy for $a_1 = a_2 \geq 2.2$.

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